**Curves in**

**1. Regular parameterised Curves**

**Definition 1.1.1.** Let be an interval. A parametrised curve in is a differentiable function . We say it is regular if for all .

*Remark.* Some things to note:

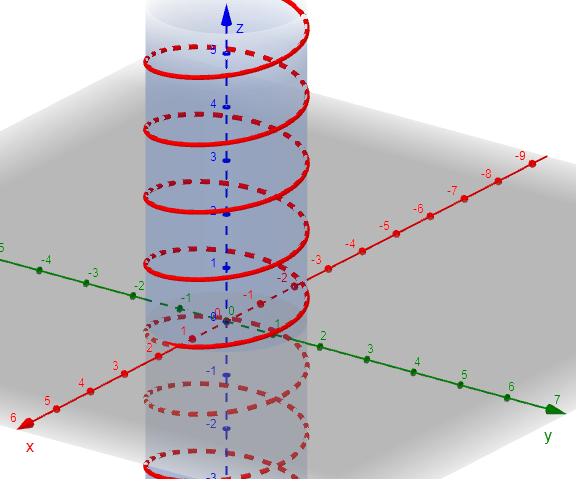
1. Differentiable will mean “of class ” (usually call smooth) is this course.
2. The image of a curve is called the trace of . If the entire trace of is continued in a single plane, we call a plane curve.
3. is called the velocity at time , and the acceleration. Thus, for any given , is a point in while is a vector, which we “think of as starting at ”.
4. need not be injective.

A black arrows with a red circle in center

AI-generated content may be incorrect.**Example 1.1.2.** Some examples of parameterised curves:

1. is a regular curve
2. *Circle and helix*. .

When , is a plane circle of radius . When s a straight line.

In general, is a helix. In all cases, is a regular curve.

Helix

A green line with black arrows

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**Cusp**

1. *Cusp*. A cusp, . Note that is not a regular curve since .

A green line with black arrows

AI-generated content may be incorrect.

1. is not regular since it is not differentiable when .

**Definition 1.1.3 (Lecture note).** Let be a parameterised curve. A change of parameters for is a differentiable bijection such that with for all ( is an interval). Then is a parametrised curve with the same trace as ; we call it a re-parametrisation. We say is orientation-preserving if for all ; otherwise, is orientation-reversing.

**Definition 1.1.3 (Textbook).** Let be two curves. A differentiable such that is called a change of parameters relating to to . The map s called orientation preserving if

*Remark.* is regular if and only if is regular.

**Definition 1.1.4.** Let be a parametrised curve. The length of between and , with , is

We say is parametrised by arc-length (or “unit-speed”) if .

**Theorem 1.1.5.** Any regular curves can be re-parametrised by arc-length.

*Proof.*

We seek such that

Let and set defined by

By the Fundamental Theorem of Calculus, is differentiable with

(greater than 0 since is regular). Set . Then is an increasing, surjective function. Then since increasing implies injective, so is bijective, so there exists an inverse . By the inverse function theorem, is differentiable and . Then is re-parametrisation, with

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*Remark.* In practice, finding is difficult.

**Example 1.1.6.** Find an arc-length re-parametrisation of

**Solution.**

So

Therefore,

is the arc-length re-parametrisation.

**Example Done.**

**1.2 The Frenet Frame**

**Definition 1.2.1.** Let be a regular curve, arc-length parametrised (i.e. for all ). The function , is called the curvature of .

Idea: measures how fast the curve pulls away from the tangent line at :

Parametrisation of tangent line at as . Then using Taylor expansion and Big-O notation, near we can write

where we know that .

**Example 1.2.2.** Find the curvature of

**Solution.**

So is indeed parametrised by arc-length. Now

**Example Done.**

**Definition 1.2.3.** Let be a regular curve parametrised by arc-length and with for all . Then

1. is the tangent vector of at .

3. is the binormal vector of at .

The mapping is called the Frenet frame of .

A diagram of a graph

AI-generated content may be incorrect.

**Theorem 1.2.4.** Let be a regular curve parametrised by arc-length and with for all . Then , is an orthonormal basis of .

*Proof.* is unit-speed parametrised, so for all . By definition of ,

We now claim that for all . Indeed, using that for all , we differentiate with respect to :

Since for all , it follows that for all .

Finally, is orthogonal to both and by definition. Moreover,

where denotes the angle between and .

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**Definition 1.2.5.** Let be a regular curve parametrised by arc-length with for all . Then

1. The plane span is called the osculating plane of .

2. The plane span is called the normal plane of .

3. The plane span is called the rectifying plane of .

**Lemma 1.2.6.** Let be a regular, unit-speed curve with for all . Then is a multiple of .

*Proof.* We need to show that and . Differentiating yields

Differentiating gives us

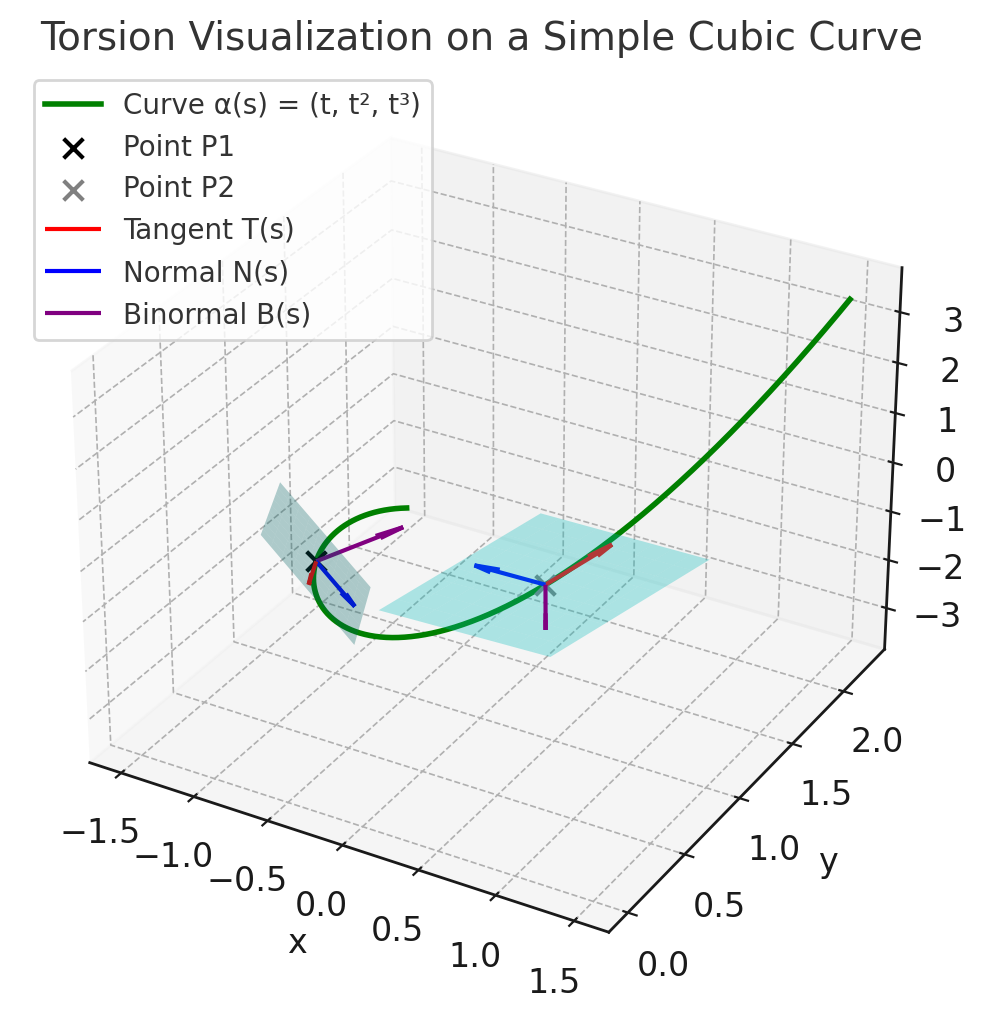
Now since is an orthonormal basis for , we can write

But from above, and , so .

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**Definition 1.2.7.** Let be a regular, unit-speed curve with for all . For each , we define the torsion of at to be the unique scalar such that

. In other words, .

*Remark.* Observe that determines the osculating plane. Thus, the torsion measures how much the curve deviated itself from its osculating plane.

**Theorem 1.2.8** (Frenet Formulas). Let ne a regular, unit-speed curve with for all . Then

which can be written more concisely as

*Proof.* The first and third equations are true by definition. For the second one, use that is an orthonormal basis for to write

Now, means that by differentiating,

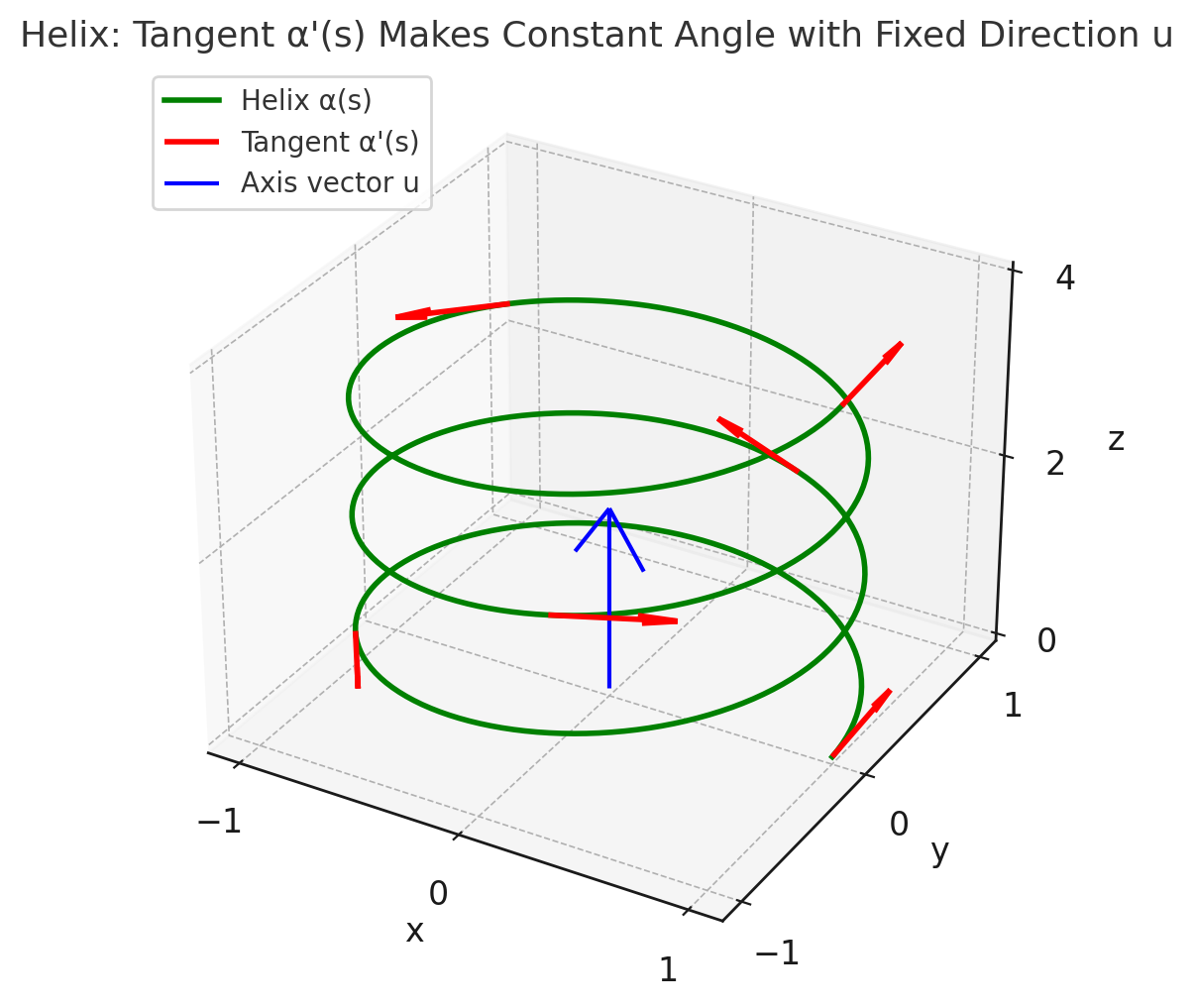
Similarly, we know that , so differentiation,

Finally, differentiating , we have

Hence,

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**Definition 1.2.9.** A regular, unit-speed curve with for all is called a helix if a unit vector such that a constant for all .



**Theorem 1.2.10.** Let be a regular, unit-speed curve with for all . Then is a helix if and only if is a constant.

*Proof.* (). Let denote the angle between and . Then by definition 1.2.9

implies that is constant, and so that . Differentiating, we get

since for all . Differentiating again, we get

from which it follows that

It remains to show that is constant:

. Suppose ; in particular, let be such that (such a exists since is surjective). Since , this implies that is orthogonal to . So lies in the span of . i.e. and say that and .

From , so using the Frenet formulas, we have ,

Thus, is indeed constant. Finally,

is constant. is therefore a helix.

**2. Curvature of Planes**

**2.1 Signed Curvature**

As we mentioned earlier the curvature of a curve is a measure of how fast it is turning. When the curve lies in a plane, we may assign a sign of plus or minus one to this measure depending on whether the curve is rotating clockwise or counterclockwise.

Thus, we arrive at a more descriptive notion of curvature for planar curves which we call signed curvature and denote by .

**Definition 2.1.1.** Given , we define a new normal , so that is an orthonormal basis for , positively oriented. Then is a multiple of , but now the coefficient has a sign. We define via to be the signed curvature of .

*Remark (formative note).*

* If the curve bends towards the left, the signed curvature is positive.
* If the curve bends toward the right, the signed curvature is negative.

