**Curves in**

**1. Regular parameterised Curves**

**Definition 1.1.1.** Let be an interval. A parametrised curve in is a differentiable function . We say it is regular if for all .

*Remark.* Some things to note:

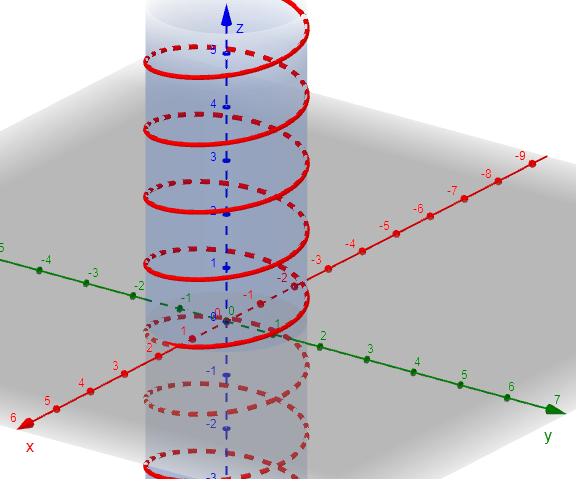
1. Differentiable will mean “of class ” (usually call smooth) is this course.
2. The image of a curve is called the trace of . If the entire trace of is continued in a single plane, we call a plane curve.
3. is called the velocity at time , and the acceleration. Thus, for any given , is a point in while is a vector, which we “think of as starting at ”.
4. need not be injective.

A black arrows with a red circle in center

AI-generated content may be incorrect.**Example 1.1.2.** Some examples of parameterised curves:

1. is a regular curve
2. *Circle and helix*. .

When , is a plane circle of radius . When s a straight line.

In general, is a helix. In all cases, is a regular curve.

Helix

A green line with black arrows

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**Cusp**

1. *Cusp*. A cusp, . Note that is not a regular curve since .

A green line with black arrows

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1. is not regular since it is not differentiable when .

**Definition 1.1.3 (Lecture note).** Let be a parameterised curve. A change of parameters for is a differentiable bijection such that with for all ( is an interval). Then is a parametrised curve with the same trace as ; we call it a re-parametrisation. We say is orientation-preserving if for all ; otherwise, is orientation-reversing.

**Definition 1.1.3 (Textbook).** Let be two curves. A differentiable such that is called a change of parameters relating to to . The map s called orientation preserving if

*Remark.* is regular if and only if is regular.

**Definition 1.1.4.** Let be a parametrised curve. The length of between and , with , is

We say is parametrised by arc-length (or “unit-speed”) if .

**Theorem 1.1.5.** Any regular curves can be re-parametrised by arc-length.

*Proof.*

We seek such that

Let and set defined by

By the Fundamental Theorem of Calculus, is differentiable with

(greater than 0 since is regular). Set . Then is an increasing, surjective function. Then since increasing implies injective, so is bijective, so there exists an inverse . By the inverse function theorem, is differentiable and . Then is re-parametrisation, with

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*Remark.* In practice, finding is difficult.

**Example 1.1.6.** Find an arc-length re-parametrisation of

**Solution.**

So

Therefore,

is the arc-length re-parametrisation.

**Example Done.**

**1.2 The Frenet Frame**

**Definition 1.2.1.** Let be a regular curve, arc-length parametrised (i.e. for all ). The function , is called the curvature of .

Idea: measures how fast the curve pulls away from the tangent line at :

Parametrisation of tangent line at as . Then using Taylor expansion and Big-O notation, near we can write

where we know that .

**Example 1.2.2.** Find the curvature of

**Solution.**

So is indeed parametrised by arc-length. Now

**Example Done.**

**Definition 1.2.3.** Let be a regular curve parametrised by arc-length and with for all . Then

1. is the tangent vector of at .

3. is the binormal vector of at .

The mapping is called the Frenet frame of .

A diagram of a graph

AI-generated content may be incorrect.

**Theorem 1.2.4.** Let be a regular curve parametrised by arc-length and with for all . Then , is an orthonormal basis of .

*Proof.* is unit-speed parametrised, so for all . By definition of ,

We now claim that for all . Indeed, using that for all , we differentiate with respect to :

Since for all , it follows that for all .

Finally, is orthogonal to both and by definition. Moreover,

where denotes the angle between and .

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**Definition 1.2.5.** Let be a regular curve parametrised by arc-length with for all . Then

1. The plane span is called the osculating plane of .

2. The plane span is called the normal plane of .

3. The plane span is called the rectifying plane of .

**Lemma 1.2.6.** Let be a regular, unit-speed curve with for all . Then is a multiple of .

*Proof.* We need to show that and . Differentiating yields

Differentiating gives us

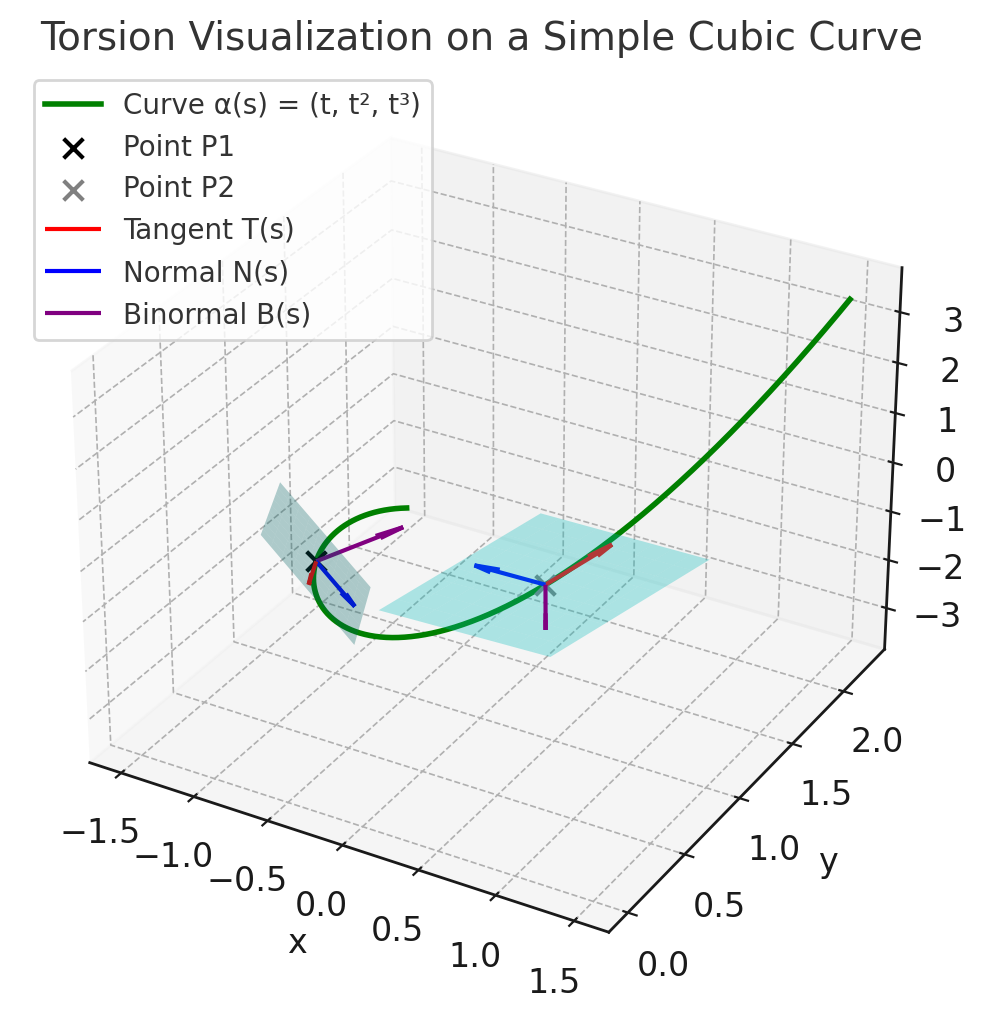
Now since is an orthonormal basis for , we can write

But from above, and , so .

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**Definition 1.2.7.** Let be a regular, unit-speed curve with for all . For each , we define the torsion of at to be the unique scalar such that

. In other words, .

*Remark.* Observe that determines the osculating plane. Thus, the torsion measures how much the curve deviated itself from its osculating plane.